# Problems for Individual Contest

**Problem 1.** Choose (1) or (2), but not both.

(1) Let $p$ be a prime number, and $k, n$ a positive integer. Count the number of subgroups of the group $(\mathbb{Z}/p\mathbb{Z})^n$ with $p^k$ elements.

(2) Classify all groups of the form $\mathbb{Z}^5/A\mathbb{Z}^n$ up to isomorphisms, where $A$ is an $5 \times n$ integer matrix and $n$ is an arbitrary positive integer.

**Problem 2.** Let $V$ be a complex vector space of dimension $n$. Let $u_1, \ldots, u_n$ be nilpotent linear endomorphisms of $V$, which pairwise commute. What can we say about their composition $u_1 \circ \cdots \circ u_n$?

**Problem 3.** (1) Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation of a finite group $G$. Let $V^G$ be the subspace of fixed points. Prove that

$$\dim V^G = \frac{1}{|G|} \sum_{G \in G} \chi(g)$$

where $\chi : G \to \mathbb{C},\ g \to \mathrm{tr}\rho(g)$ is the character.

(2) Consider the graded ring $S = \mathbb{C}[x_1, \cdots, x_n] = \oplus_{d \geq 0} S_d$. Let $G \subset \mathrm{GL}(n)$ be any finite subgroup with the induced action on $S$. We have $S^G = \oplus_{d \geq 0} S^G \cap S_d$. Prove that

$$\sum_{d \geq 0} (\dim S^G \cap S_d) t^d = \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det(I_n - tA)}.$$